Critical Orbit Structure of $f_c(z) = z^2 + c$ over $\mathbb{C}_p$

Cara E. Mullen
Department of Mathematics, University of Illinois at Chicago

Working with L. DeMarco
Professor at Northwestern University

Question

Fix a prime $p > 2$ and consider $f_c(z) = z^2 + c$ with $c \in \mathbb{C}_p$. What are the different possibilities for the structure of the orbit of 0, under iteration of f? A full classification of finite critical orbit trees is known in the complex setting; we will study some of the variations over $\mathbb{C}_p$ by looking at the critical portrait of the reduction map $\bar{f}_c$ (over the residue field $k = \mathcal{O}_{\mathbb{C}_p}/\mathfrak{m}_p = \overline{\mathbb{F}}_p$).

Motivation

Hubbard trees classify hyperbolic components of the Mandelbrot set in the complex setting. What are the analogous trees in Berkovich space, in the p-adic setting?

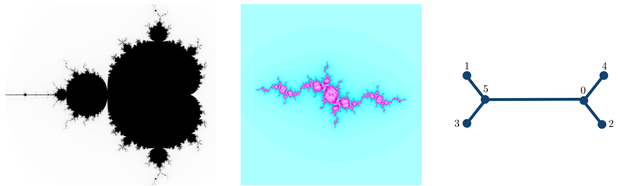


Figure 1: The complex Mandelbrot set; the Julia set for a period-6 critical orbit in the $1/2$ limb; and its Hubbard tree.

First, we look at the classical Mandelbrot set that gives the picture above:

$$\mathcal{M} = \{c \in \mathbb{C} : \text{ critical orbit of } f_c(z) = z^2 + c \text{ is finite}\}$$

And the corresponding set in the p-adic setting:

$$\mathcal{M}_p = \{c \in \mathbb{C}_p : \text{ critical orbit of } f_c(z) = z^2 + c \text{ is bounded}\},$$

which is actually the rather bland unit disk $D = \{\alpha \in \mathbb{C}_p : |\alpha| \leq 1\}$.

Given the rich structure of the space $\mathbb{C}_p$, however, $\mathcal{M}_p$ is more than "just a disk": it is an infinitely branching tree with many interesting features. These features lead to some unexpected patterns in the structure of the critical orbit for the post-critically bounded (PCB) polynomials over $\mathbb{C}_p$.

Setting

Let $\mathbb{C}_p$ be the algebraically closed (metric) completion of $\mathbb{Q}_p$. Let $|\cdot|_v$ be a non-Archimedean absolute value which extends a p-adic absolute value. It satisfies the strong triangle inequality:

$$|\alpha + \beta|_v \leq \max\{|\alpha|_v, |\beta|_v\} \text{ for all } \alpha, \beta \in \mathbb{C}_p.$$

Define

$$\begin{aligned} D &= \{\alpha \in \mathbb{C}_p : |\alpha|_v \leq 1\}, \text{ the valuation ring of } \mathbb{C}_p, \text{ the closed unit disk.} \\ U &= \{\alpha \in \mathbb{C}_p : |\alpha|_v < 1\}, \text{ the maximal ideal of } D, \text{ the open unit disk.} \\ \overline{\mathbb{F}}_p &= D/U, \text{ the residue field of } D. \\ \mathbb{Z}_p &= \{z \in \mathbb{Q}_p : z = \sum_{i=0}^{\infty} a_i p^i, \text{ where } a_i \in \{0, \ldots, p-1\}\} \end{aligned}$$

Note $|\cdot|_v = \pi^{-v(\alpha)}$ for some $\pi > 1$, where $v(\alpha) = \mathrm{ord}_p(\alpha) =$ the exponent of the highest power of p dividing α.

Main Results

Definition: A point α has **orbit type** $(\mathbf{m}, \mathbf{n})$ if m and n are the least integers such that $f^{m+n}(\alpha) = f^m(\alpha)$. Then m is the tail length and n is the cycle length of the orbit of α.

The following theorem applies to all polynomials of the form $f_c(z) = z^2 + c$ over $\mathbb{Z}_p$, both with finite critical orbit and infinite critical orbit.

Theorem: Critical Orbit Type

For $f_c(z) = z^2 + c$, $c \in \mathbb{Z}_p$, $p > 2$, we have the following:

1. Periodic Reduction:
   If 0 has orbit type $(0, n)$ (mod p), then 0 has orbit type $(s(k), n)$ (mod p^k) for all k, where $s(k)$ is a constant that depends on k.
2. Pre-periodic Reduction:
   If 0 has orbit type (m, n) (mod p), then 0 has orbit type $(m, t(k) \cdot n)$ (mod p^k) for all k, where $t(k) = t \cdot p^i$ for $t < p$ and $i < k$.

Key ingredients of the proof:

1. Lemma: If 0 is periodic of exact period n under iteration of $f_c(z)$ (mod p), then there exists an attracting n-cycle of for $f_c(z)$.
   The proof of this lemma relies on Hensel's Lemma. See also Corollary 1.5 in [2].
2. In local coordinates, away from the critical point, the reduced iterates of f are linear transformations, which act bijectively on the periodic cycles.

Remark: The patterns that arise in infinite critical orbits over $\mathbb{Z}_p$ give insight about possible structures of finite critical orbits over $\mathbb{C}_p$. It is known that if 0 is periodic of period n over $\mathbb{C}_p$, then 0 will have exact period n under the reduced map over $\overline{\mathbb{F}}_p$. If 0 is strictly pre-periodic over $\mathbb{C}_p$, we have the following:

Conjecture: Pre-periodic Critical Orbit

If 0 is pre-periodic over $\mathbb{C}_p$ of orbit type (m, n), it will follow the same orbit structure as that of a map with infinite critical orbit and reduced orbit type (m, n'), with $n = n' \cdot r \cdot p^i$ for n a root of unity (mod p) of order r.

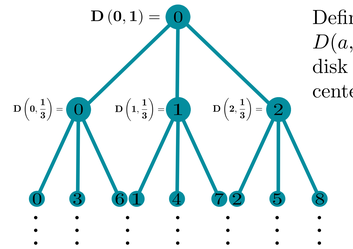
Extension

Once the patterns in the parameter space and in the critical orbit structure are fully understood, a natural question to ask is: do the same patterns exist in $\mathcal{M}_p$ and the Julia set corresponding to a polynomial with parameter in $\mathcal{M}_p$?

This correspondence exists in the complex setting (see [3]), and Anderson has shown in [1] that it holds for at least one example in the 2-adic setting, degree 3.

Example: Dynamics over $\mathbb{Z}_3$

$\mathbb{Z}_3$ may be viewed as a tree inside the closed unit disk of $\mathbb{C}_3$, so any parameter in $\mathbb{Z}_3$ corresponds to a PCB polynomial. A visual of the top of the $\mathbb{Z}_3$ tree:



Definition: $D(a, r)$ is the closed disk of radius r centered at a.

Structure of $\mathcal{M}_3$ over $\mathbb{Z}_3$:

- $D(0, 3^{-1})$ = Hyperbolic Component 1: parameters with an attracting fixed point.
- $D(2, 3^{-1})$ = Hyperbolic Component 2: parameters with an attracting 2-cycle.
- $D(1, 3^{-1})$ contains parameters with strictly pre-periodic critical points.

In $D(1, 3^{-1}) \cap \mathbb{Z}_3$, there are only 2 possible finite critical orbits: (2,1) and (2,3). The other parameters have infinite (but bounded) critical orbits, and those orbits grow in a fixed pattern. Follow $c \equiv -2$, which has finite critical orbit type (2,1):

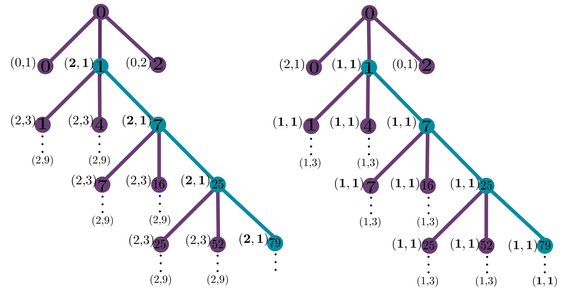


Figure 2: The parameter space near $c \equiv -2$ with corresponding critical orbit type at each vertex on the left, and the Julia set of $f(z) = z^2 - 2$ near $z = -2$ in $\mathbb{Z}_3$ with corresponding orbit type of each vertex on the right.

References

[1] J. Anderson. *p-adic Analogues of the Mandelbrot Set.* PhD thesis, Brown University, 2010.

[2] R. Jones. *Galois Martingales and the Hyperbolic Subset of the p-adic Mandelbrot Set.* PhD thesis, Brown University, 2005.

[3] T. Lei. Similarity Between the Mandelbrot Set and Julia Sets. *Comm. Math. Phys.*, 134(3):587-617, 1990.

[4] J. Silverman. *The Arithmetic of Dynamical Systems.* Springer, New York, 2007.